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Finite-Size Effects in a Field-Theoretic Model with Long-Range Exchange Interaction

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We present a systematic approach to the calculation of finite-size (FS) effects for an $O(n)$ field-theoretic model with both short-range (SR) and long-range (LR) exchange interactions. The LR exchange interaction decays at large distances as $1/r^{d+2-2\alpha}$, $\alpha \to 0^+$. Renormalization group calculations in $d = d_u - \varepsilon$ are performed for a system with a fully finite (block) geometry under periodic boundary conditions. We calculate the FS shift of the critical temperature and the FS renormalized coupling constant of the model to one-loop order. The universal scaling variable is obtained and the FS scaling hypothesis is verified.

KEY WORDS: Finite-size scaling, long-range interactions, ε-expansion.

1. INTRODUCTION

Finite-size scaling (FSS) theory for a system with long-range interactions decaying at large distances r as $r^{-d-\sigma}$ (d is the space dimensioality and $0 < \sigma \leqslant 2$ is a parameter) was first developed by Fisher and Privman.[1] Recently, this problem was considered in refs. 2–5. However, all the efforts in this direction have been concentrated on the mean spherical model, which is probably due to its relative simplicity and remarkable property to be exactly soluble. It is of considerable theoretical interest to understand how the FSS behaves in more realistic cases, for example, in a φ^4-lattice model.

The application of field-theoretic methods to the FSS theory has been initiated by Brézin.[6] Brézin and Zinn-Justin[7] and Rudnick *et al.*[8] showed how to study the size-dependent universal effects in an ε-expansion in the neighborhood of the upper critical dimensionality $d_u = 4$ for the

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φ^4-field theory. The main idea in this approach is to represent the Hamiltonian as consisting of two parts: the $k=0$ mode as an effective free part and the $k \neq 0$ modes as a perturbation to it. This requires the existence of a well-developed perturbation theory as an indispensable condition as in the case of the common bulk φ^4-field theory.

Recently it was pointed out by Honkonen and Nalimov[9] that the model with both SR and LR exchange may be treated by the renormalization group procedure in the minimal subtraction scheme. The LR interaction in ref. 9 is of the special type $\sigma = 2 - 2\alpha$, when $\alpha \to 0^+$. This permits one to treat the LR term as an interaction and to construct a diagrammatic expansion in α. The results obtained in ref. 9 allow us to follow the ideology of refs. 7 and 8 and to present a systematic approach to the calculation of the FS effects for the $O(n)$ field-theoretic model of this type.

In Section 2 the model is defined and some results for the bulk system which we shall use in the following sections are summarized. In Sections 3 and 4 the expressions for the renormalized constants of the model are derived up to one-loop order. The finite-size scaling hypothesis is analyzed in Section 5.

2. MODEL

We consider the following Landau–Ginzburg Hamiltonian:

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_I \tag{1}$$

where $\mathscr{H} \equiv -H/T$, $k_B = \hbar = 1$,

$$\mathscr{H}_0 = \sum_{i=1}^{n} \sum_{\mathbf{k}} \left(\frac{1}{2} k^2 - \frac{b_0}{2} k^\sigma + t_0 \right) \varphi_i^2(\mathbf{k}) \tag{1a}$$

is the free part of $\mathscr{H}$, and

$$\mathscr{H}_I = \frac{\lambda}{4!\, V} \sum_{i,j=1}^{n} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \varphi_i(\mathbf{k}_1)\, \varphi_i(\mathbf{k}_2)\, \varphi_j(\mathbf{k}_3)\, \varphi_j(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3) \tag{1b}$$

is the φ^4-interaction between fluctuations.

The first and the second terms in Eq. (1a) correspond to SR and LR exchange interactions, respectively. $\varphi_i(\mathbf{k})$ is a scalar n-component field and $t_0 = a(T - T_0)$, where T_0 is the bare critical temperature for the bulk system.

We suggest that $\sigma = 2 - 2\alpha$ with $\alpha \sim \varepsilon$, where $\varepsilon = d_u - d = 4 - 4\alpha - d$ (d_u is the upper critical dimensionality).

A tractable diagrammatic expansion to all orders in perturbation theory has been constructed in ref. 9, where the LR term is regarded as an interaction, and dimensional regularization with minimal subtraction is used.

Counterterms due to renormalization lead to the replacements[10]

$$t \to tZ_{\varphi^2} \tag{2a}$$

$$g \to gZ_1 \tag{2b}$$

$$b \to bZ_{\varphi} \tag{2c}$$

In Eqs. (2), $t = a(T - T_c(\infty))$, where $T_c(\infty)$ is the renormalized critical temperature of the bulk system, g is the renormalized coupling constant, and b is the renormalized exchange constant. Everywhere below we shall use dimensionless constants with length scale set to unity.

In one-loop order we have

$$Z_{\varphi^2} \cong 1 + \sum_{l=0}^{\infty} (-b)^l \hat{g} \left(\frac{n+2}{6}\right)(l+1)\frac{1}{\varepsilon + 2l\alpha + 4\alpha} \tag{3a}$$

$$Z_1 \cong 1 + \sum_{l=0}^{\infty} (-b)^l \hat{g} \left(\frac{n+8}{6}\right)(l+1)\frac{1}{\varepsilon + 2l\alpha + 4\alpha} \tag{3b}$$

and

$$Z_{\varphi} \cong 1 + \sum_{l=0}^{\infty} (-b)^l \hat{g}^2 \left(\frac{n+2}{72}\right)\binom{l+2}{l}\frac{1}{2\varepsilon + 2l\alpha + 8\alpha} \tag{3c}$$

In Eqs. (3)

$$\hat{g} = g\,\frac{2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} = \frac{g}{8\pi^2}\left[1 - \frac{\varepsilon}{2}(C - 1 - \ln 4\pi) + O(\varepsilon^2)\right] \tag{4}$$

Here C and $\Gamma(\cdot)$ are Euler's constant and the gamma function. Equations (3b) and (3c) have been obtained in ref. 9. We shall use also the following results obtained in ref. 9:

(i) In the presence of LR exchange interaction the Wilson gamma functions $\gamma_{\varphi_{\mathrm{LR}}}$ and $\gamma_{1_{\mathrm{LR}}}$ are given by the same functions as in the SR case, replacing only the argument $\hat{g}$ of the SR case by $\hat{g}(1+b)^{-2}$. For example,

$$\gamma_{1_{\mathrm{LR}}}(b, \hat{g}) = \gamma_{1_{\mathrm{SR}}}[\hat{g}(1+b)^{-2}] \tag{5}$$

(ii) The LR fixed-point equation for the effective parameter $u = \hat{g}(1+b)^{-2}$ is

$$\gamma_{1_{\mathrm{SR}}}(u^*) = \varepsilon \tag{6}$$

For our purposes we shall use the one-loop order for the Wilson function $\gamma_{1_{SR}}(u)$:

$$\gamma^{(1)}_{1_{SR}} = \frac{n+8}{6} u \tag{7}$$

Because we shall work up to first order in ε, it will suffice to replace Z_φ by 1 in Eq. (2c).

3. FINITE-SIZE SHIFT OF T_c

The finite-size shift of T_c is given by the following expression[7]:

$$\tilde{t} = t_R + t_L \tag{8}$$

with

$$t_R = tZ_{\varphi^2} \tag{8a}$$

and

$$t_L = \left(\frac{n+2}{6}\right) g \frac{1}{L^d} \sum_{l=0}^{\infty} \sum_q{}' (-b)^l \frac{q^{2l(1-\alpha)}}{(q^2+t)^{1+l}} \tag{8b}$$

The term t_R comes from the one-loop counterterm [see Eqs. (2a) and (3a)] and the term t_L is the finite-size correction. The expression for t_L is obtained using the diagrammatic technique developed in ref. 9.

Equation (8) contains the result of ref. 7 (setting formally $l=0$) for the SR case.

The prime in the d-fold sum in Eq. (8) denotes that the term with a zero summation index has been omitted. In order to reduce the problem of evaluating the asymptotic behavior of the sum over q to the corresponding one-dimensional sum, we shall use the following identity:

$$\frac{q^{2l(1-\alpha)}}{(q^2+t)^{1+l}} = \frac{1}{\Gamma(1+l\alpha)} \int_0^\infty dx\, e^{-q^2 x} x^{l\alpha}\, {}_1F_1(1+l; 1+l\alpha; -tx) \tag{9}$$

In Eq. (9), ${}_1F_1$ is the degenerate hypergeometric function[11] (Kumer's function) defined as

$${}_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} \tag{10}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ and $(a)_0 = 1$. Identity (9) follows directly regarding the function $x^{l\alpha}\, {}_1F_1(1+l; 1+l\alpha; -tx)$ as a Laplace original of $q^{2l(1-\alpha)}/(q^2+t)^{1+l}$.

Now, using the identity (9) for the finite-size term t_L in the rhs of Eq. (8), we obtain

$$t_L = \sum_{l=0}^{\infty} (-b)^l g\left(\frac{n+2}{6}\right) \frac{t^{d/2-1-l\alpha}}{(4\pi)^{d/2}} \frac{\Gamma(1+l\alpha-d/2)\,\Gamma(l-l\alpha+d/2)}{\Gamma(1+l)\,\Gamma(d/2)}$$
$$+ \sum_{l=0}^{\infty} (-b)^l g\left(\frac{n+2}{6}\right) \frac{1}{\Gamma(1+l\alpha)} \frac{L^{2-d}}{4\pi^2}\left(\frac{L^2}{4\pi}\right)^{l\alpha}$$
$$\times \int_0^{\infty} dx\left[A^d(x)-1-\left(\frac{\pi}{x}\right)^{d/2}\right] x^{l\alpha}\, {}_1F_1\left(1+l; 1+l\alpha; -\frac{L^2 t}{4\pi^2}x\right) \tag{11}$$

where

$$A(x) = \sum_{m=-\infty}^{\infty} e^{-xm^2} \tag{12}$$

The first term in Eq. (11) is obtained replacing the function $A^{d/2}(x)$ by its asymptotic expression $(\pi/x)^{d/2}$, using

$$\int_0^{\infty} t^{b-1}\, {}_1F_1(a; c; -t)\, dt = \frac{\Gamma(b)\,\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}$$

and continuing analytically the obtained result for $\operatorname{Re} d > 2 + 2l\alpha$. From Eqs. (8a) and (3a), it follows that t_R has an $(\varepsilon + 2l\alpha + 4\alpha)^{-1}$ pole. This pole cancels the pole of the Γ-function in the first term in t_L.

Expanding all functions in t_L near the upper critical dimensionality up to order ε^1 and α^1 and after some algebra, for Eq. (8) we obtain the result

$$\tilde{t} = \sum_{l=0}^{\infty} (-b)^l t\hat{g}\left(\frac{n+2}{12}\right)(l+1)[\ln t + C - 1 + \psi(l+2)]$$
$$+ \sum_{l=0}^{\infty} (-b)^l g\left(\frac{n+2}{6}\right)\frac{L^{2-d}}{4\pi^2}\int_0^{\infty} dx\left[A^d(x)-1-\left(\frac{\pi}{x}\right)^{d/2}\right]$$
$$\times\, {}_1F_1\left(1+l; 1; -\frac{L^2 t}{4\pi^2}x\right) + O(g^{\alpha}) \tag{13}$$

where $\psi(\cdot)$ is the digamma function.

The summation over l in Eq. (13) can be performed if we use the well-known integral representation

$$\psi(l+2) = -C + \int_0^1 \frac{1-x^{l+1}}{1-x}\, dx$$

and the formula[11]

$$\sum_{l=0}^{\infty} (-b)^l L_l^0(z) = \frac{e^{z[b/(1+b)]}}{1+b} \tag{14}$$

where $L_l^0(z) = e^{z}\, {}_1F_1(1+l;\, 1;\, -z)$ is the Laguerre polynomial.

Finally, we obtain the following expression for the finite-size shift of T_c:

$$\begin{aligned} \frac{\tilde{t}}{1+b} &= \frac{t}{1+b} + \frac{n+2}{12} u \frac{t}{1+b} \ln\left[\frac{t}{1+b}\right] \\ &+ u\left(\frac{n+2}{3}\right) L^{-2} \int_0^{\infty} dx \left[A^4(x) - 1 - \left(\frac{\pi}{x}\right)^2\right] \\ &\times \exp\left[-\frac{L^2}{4\pi^2}\frac{t}{1+b} x\right] + O(u\alpha) \end{aligned} \tag{15}$$

This equation generalizes the result of Brézin and Zinn-Justin for the pure SR case, renormalizing the temperature by the rule $t \to t/(1+b)$.

4. RENORMALIZATION OF THE COUPLING CONSTANT

In a similar way,

$$\tilde{g} = g_R + g_L \tag{16}$$

where the one-loop counterterm is

$$g_R = g Z_1 \tag{16a}$$

and the one-loop finite-size correction is:

$$g_L = -g^2 \left(\frac{n+8}{6}\right) \frac{1}{L^d} \sum_{l=0}^{\infty} \sum_q{}' (-b)^l (l+1) \frac{q^{2l(1-\alpha)}}{(q^2+t)^{l+2}} \tag{16b}$$

The calculations here are analogous to those in the previous section, since

$$\sum_q{}' \frac{q^{2l(1-\alpha)}}{(q^2+t)^{l+2}} = -\frac{1}{l+1}\frac{\partial}{\partial t}\sum_q{}' \frac{q^{2l(1-\alpha)}}{(q^2+t)^{l+1}}$$

Taking first the sum over q and then differentiating the result over t, we find that

$$\tilde{u} = \frac{\tilde{g}}{8\pi^2(1+b)^2}$$

$$= u + \frac{n+8}{12} u^2 \left[\ln\left(\frac{t}{1+b}\right) + 1 \right]$$

$$- u^2 \left(\frac{n+8}{12}\right) \frac{1}{\pi^2} \int_0^\infty x\, dx \left[A^4(x) - 1 - \left(\frac{\pi}{x}\right)^2 \right]$$

$$\times \exp\left[-\frac{L^2}{4\pi^2} \frac{t}{1+b} x \right] + O(u^2\alpha) \tag{17}$$

From (17) it follows that the consideration of the LR term in Eq. (1a) effectively results in the replacement $\hat{g} \to u = \hat{g}/(1+b)^2$ and $t \to t/(1+b)$ in the corresponding formula of Brézin and Zinn-Justin.[7]

5. FINITE-SIZE SCALING

Let us introduce the total spin per unit volume

$$\varphi = \frac{1}{V} \int_V d^d\mathbf{x}\, \varphi(\mathbf{x}) \tag{18}$$

Then for the moments $M_{2p} = \langle (\varphi^2)^p \rangle$, where the average is taken at the tree level, we have

$$M_{2p} = (\lambda L^d)^{-p/2} f_{2p}(tL^{d/2}\lambda^{-1/2}) \tag{19}$$

where $f_{2p}(\cdot)$ is a known function.[7]

The susceptibility χ is proportional to M_2, $\chi \sim L^d M_2$. Then

$$\chi(t, L) = L^{d/2} f(tL^{d/2}) \tag{20}$$

Since the correlation length is $\xi \sim t^{-\nu}$ (where $\nu = 1/\sigma$ for $d > d_u$), the susceptibility may be written in the form

$$\chi(t, L) = t^{-1} \tilde{f}(L^{(d-2\sigma)/2\sigma} \times L/\xi) \tag{21}$$

From Eq. (21) one can see that above the upper critical dimensionality $d_u = 4 - 4\alpha$ the usual finite-size scaling does not hold.

For testing the FSS hypothesis in the vicinity below d_u let us consider the scaling variable $z = \tilde{t} L^{d/2} \tilde{g}^{-1/2}$ [see also Eq. (19)] at the fixed point. This variable is very important in investigations of the FS effects in critical statics[7] and critical dynamics.[12,13] We have

$$z|_{\text{f.p.}} = \frac{\tilde{t}}{1+b} L^{d/2} \frac{\tilde{u}^{1/2}}{(8\pi)^{1/2}}\bigg|_{\text{f.p.}}$$

$$\approx \frac{(u^*)^{-1/2}}{(8\pi)^{1/2}} \left\{ y - \frac{1}{4}\varepsilon y + \frac{n-4}{4(n+8)}\varepsilon y \ln y \right.$$

$$+ \frac{2(n+2)}{n+8}\varepsilon \int_0^\infty dx \left[A^4(x) - 1 - \left(\frac{\pi}{x}\right)^2 \right] e^{-(y/4\pi^2)x}$$

$$\left. + \frac{y}{4\pi^2}\varepsilon \int_0^\infty dx\, x \left[A^4(x) - 1 - \left(\frac{\pi}{x}\right)^2 \right] e^{-(y/4\pi^2)x} \right\} \tag{22}$$

To obtain the above expression we used $u^* = 6\varepsilon/(n+8)$, $\nu^{-1} = 2 - \varepsilon(n+2)/(n+8) - 2\alpha$, and the fact that up to order ε^1 the terms proportional to $\ln L$ cancel. For convenience in Eq. (22) the characteristic variable $y = [t/(1+b)]\, L^{1/\nu}$ is introduced. The result (22) verifies the FSS hypothesis.

The scaling variable z may be expressed as an analytic function of the variable y. For this purpose we shall use the expansion[5,12]

$$\int_0^\infty dx \left[A^4(x) - 1 - \left(\frac{\pi}{x}\right)^2 \right] e^{-(y/4\pi^2)x}$$

$$= a_0 + a_1 y - \frac{y}{4} \ln y + a_2 y^2 + O(y^3) \tag{23}$$

The first two coefficients in Eq. (23) are

$$a_0 = \int_0^\infty dx \left[A^4(x) - 1 - \left(\frac{\pi}{x}\right)^2 \right] \approx -1.77\pi \tag{24a}$$

and

$$a_1 = -\frac{1}{4}(1 + \tilde{C}), \qquad \tilde{C} = \text{const (see ref. 12)} \tag{24b}$$

From Eq. (23) one obtains for the finite-size shift of T_c [see Eq. (15)] and for the scaling variable z [see Eq. (22)] the following expressions, respectively:

$$\tilde{t} \approx t\left(1 + \frac{g}{(1+b)^2}\frac{n+2}{3}a_1 - \frac{n+2}{6}\frac{g}{(1+b)^2}\ln L\right) + \frac{g}{(1+b)^2}\frac{n+2}{3}L^{-2}a_0 \tag{25}$$

and

$$z|_{\text{f.p.}} \approx \frac{(u^*)^{-1/2}}{(8\pi)^{1/2}} \{ y + \varepsilon[a_0(n) + a_1(n)\, y + a_2(n)\, y^2 + \cdots] \} \tag{26}$$

with

$$a_l(n) = \left[\frac{2(n+2)}{n+8} - l\right] a_l \tag{27}$$

Equation (26) gives us a systematic approach to obtaining the variable z in powers of y, when $y \ll 1$.

6. CONCLUDING REMARKS

Comparing the derived equations (15) and (17) with the corresponding results for the pure SR case,[7] one can see that the main difference consists in rescaling the temperature t $[t \to t/(1+b)]$ and the constants g and b $[g \to g/(1+b)^2;\ b \to b]$. This is a consequence of the special type of LR interaction ($\alpha \sim \varepsilon$) in the bulk system.

The crucial point in the approach proposed in ref. 7 is the cancellation of the pole originating in the one-loop counterterm with the pole of the finite-size correction term. The origin of the last is connected with the fact that the discrete sums over q in the field theory are extended to infinity (i.e., the zero lattice spacing is taken). It is known[9] that the pole of the counterterm for the SR case is removed from its value ε^{-1} to the value $(\varepsilon + 2l\alpha + 4\alpha)^{-1}$, which corresponds to the LR interaction. We demonstrated that in the case under consideration this pole also cancels the pole of the finite-size term.

The scaling variable z entering as an argument of the momenta M_{2p} in Eq. (19) is calculated at $d < d_u$ at the one-loop order. In our case, due to the obtained form of the rescaled variables t and g, one concludes that z is a universal quantity with respect to the presence of the LR interaction term [see Eq. (25)].

Our analysis shows that the renormalized values of the temperature and the coupling constant are continuous functions of the parameter α when $\alpha \to 0$. This holds for both cases of infinite and finite systems. In the latter case, the result was established to first order of the perturbation theory over g.

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